ON REMOVABLE SINGULARITIES OF MEROMORPHIC CORRESPONDENCES

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Given a domain U in $\mathbb{C}^n$, $n > 1$, a *meromorphic correspondence* is an irreducible complex analytic subset $A \subset U \times \mathbb{CP}^N$, $N \geq 1$, of dimension n with proper surjective projection $\pi : A \to U$. The complex analytic subset $S \subset U$ defined by

$$S = \{z \in U : \dim \pi^{-1}(z) > 0\} \tag{1}$$

is called the *fundamental set* or the *indeterminacy locus* of A. Note that $\dim S \leq n-2$. Outside S, the set A defines a locally proper holomorphic correspondence given by $\pi' \circ \pi^{-1}$, where $\pi' : A \to \mathbb{CP}^N$ is the other coordinate projection. The set A is called a *meromorphic map* if $\pi|_{A \setminus S}$ is biholomorphic. For more properties of meromorphic correspondences and maps see, e.g., [2] or [3]. The goal of this note is to prove the following result.

Theorem 1. *Suppose that $M \subset U \subset \mathbb{C}^n$ is a C^1-smooth real hypersurface, and $K \subset \mathbb{CP}^N$ is a compact subset that does not contain any germs of nonconstant complex curves. Assume that for any $z \in M \setminus S$, $\pi' \circ \pi^{-1}(z) \subset K$. Then $M \cap S = \varnothing$.*

It follows from the theorem that there is a neighbourhood of M in U such that $\pi' \circ \pi^{-1}$ is, in fact, a holomorphic correspondence (or a map, if A is a meromorphic map) sending M into K, and so it can be viewed as a result on removable singularities of meromorphic correspondences and maps. Note that Theorem 1 is a local result and can be applied in a neighbourhood of any point on $S \cap M$, and that no assumption is required on the Levi form of M (if M is C^2-smooth).

Theorem 1 above generalizes a well-known lemma of Pinchuk [4] (see also [6]) stating that a meromorphic map from a real analytic strictly pseudoconvex hypersurface M with the image of $M \setminus S$ contained in the sphere $S^{2N-1} \subset \mathbb{C}^N$ for $N \geq n$ extends holomorphically everywhere on M.

A theorem of Webster [7] states that if f is a germ at zero of a biholomorphic map sending a real algebraic hypersurface M in $\mathbb{C}^n$, Levi-nondegenerate at zero, into another such hypersurface M', then f is necessarily algebraic, that is, the graph of f is contained in an algebraic subset $A \subset \mathbb{CP}^n \times \mathbb{CP}^n$ of dimension n. This gives extension of f as a meromorphic correspondence. The extension of f is holomorphic almost everywhere on M except possibly points on M that have positive-dimensional fibres in A. Now it follows from Theorem 1 that if M' is compact and does not contain any complex curves, then the extension is holomorphic everywhere on M. A similar conclusion can be drawn in many other situations when the graph of a map or correspondence extends as an analytic set of the same dimension, e.g., in [1, Cor. 1.4]. We also remark that Theorem 1 ensures that various extensions obtained in [5] are holomorphic.

Proof of Theorem 1. The proof is by induction on n, the dimension of U. Suppose first that $n = 2$. Then S is discrete, and we may assume without loss of generality that $b \in M$ is the only point in U of $S \cap M$. Then $\pi^{-1}(b)$ is a possibly reducible one dimensional complex analytic set and we will refer to it as the 'vertical' component over b. Let H_bM be the complex tangent line to M at b.

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Now $((H_bM \cap U) \times \mathbb{CP}^N) \cap A$ is a one dimensional complex analytic subset of A that consists of the vertical component $\pi^{-1}(b)$ and the 'horizontal' part, namely

$$X = \overline{\pi^{-1}((H_bM \cap U) \setminus \{b\})}.$$

X is a one dimensional complex analytic subset of A and is the union of those irreducible components of $((H_bM \cap U) \times \mathbb{CP}^N) \cap A$ that are distinct from $\pi^{-1}(b)$. Note that each irreducible component of X properly surjects onto H_bM and that the fibre in X over b is discrete. Let (b, b') be an arbitrary point in the fibre $\pi^{-1}(b)$ in A that does not belong to X. We claim that $b' \in K$.

For the proof of the claim observe that since the projection π is proper and surjective onto U and the set A is irreducible, no neighbourhood of (b, b') in A can be mapped by π into $\{b\}$. Hence, there exists a sequence of points $\{(b^j, b'^j)\} \subset A$ converging to (b, b') with $b^j \neq b$. Let L_j be a complex line passing through the points b and b^j. After passing to a subsequence we may assume that the L_j's converge to a complex line L_0 passing through b. Let

$$E_j = \overline{\pi^{-1}((L_j \cap U) \setminus \{b\})}.$$

Again, each E_j is a complex analytic subset of A of dimension one and by construction consists only of the 'horizontal' part of $((L_j \cap U) \times \mathbb{CP}^N) \cap A$, i.e., each irreducible component of E_j is distinct from $\pi^{-1}(b)$. Let F_j be the union of all irreducible components of E_j that pass though (b^j, b'^j). Note that each component in F_j projects surjectively onto $L_j \cap U$ and that all fibres in F_j over $L_j \cap U$ are discrete. Finally, after passing to a subsequence, if needed, let $F_0 = \lim F_j$. By construction, $\pi(F_0) \subset L_0$. Furthermore, $\pi(F_0) = L_0 \cap U$. Indeed, if $\pi(F_0) \neq L_0 \cap U$, then by properness of π, $\pi(F_0) = \{b\}$. But each sphere in U centred at b of sufficiently small radius has a nonempty intersection with $\pi(F_j)$, and hence will have a point from $\pi(F_0)$. Thus F_0 contains an irreducible component $\tilde{F}_0$ which contains the point (b, b') with $\pi(\tilde{F}_0) = L_0 \cap U$. In particular, $\tilde{F}_0$ is a 'horizontal' component. The fibre of b in $\tilde{F}_0$ is discrete, and so $\tilde{F}_0$ defines a holomorphic correspondence $F : L_0 \to \mathbb{CP}^N$.

Since $(b, b') \notin X$, it follows that $L_0 \neq H_bM$, so the intersection at b of L_0 with M is transverse, and there exists a curve $\gamma \subset L_0 \cap M$ passing through b. Then the cluster set of F along γ contains the point (b, b'). But this implies that the cluster set of the correspondence $\pi' \circ \pi^{-1}$ restricted to $M \setminus \{b\}$ contains (b, b'). By assumption, on $M \setminus \{b\}$ this correspondence attains values in K only, and we conclude that $b' \in K$. This proves the claim.

It follows from the claim that except a discrete set (which are exactly the points in X over b), the fibre of b in A projects under π' to K. Since $\pi^{-1}(b) \subset A$ is closed, it follows that $\pi' \circ \pi^{-1}(b) \subset K$. On the other hand, $\pi' \circ \pi^{-1}(b) \subset K$ is a union of complex analytic sets, but since K does not contain any germs of positive dimensional curves we conclude that $\pi^{-1}(b)$ must be discrete. Hence, $b \notin S$. Note that it follows that $\pi' \circ \pi^{-1}(b) \subset K$.

For the induction argument, suppose that the result is proved for $n-1$. Now given a meromorphic correspondence $A \subset U \times \mathbb{CP}^N$ for $U \subset \mathbb{C}^n$, let $b \in M \cap S$ be arbitrary. As in the previous argument it is enough to show that $\pi' \circ \pi^{-1}(b) \subset K$ to prove the theorem.

Let (b, b') be an arbitrary point in the fibre of b. Let $\{(b^j, b'^j)\}$ be a sequence of points in A converging to (b, b') with $b^j \notin S$. For each j, choose some complex hyperplane L_j passing through points b, b^j, and containing the complex normal to M at b. By passing to a subsequence we may assume that the L_j's converge to a complex hyperplane L_0 not equal to H_bM. Then L_0 contains the complex normal to M at b and hence L_0 intersects M transversally. The set

$$A_0 = \overline{\pi^{-1}((L_0 \cap U) \setminus S)}$$

is complex analytic; its dimension is $n-1$ because generic fibres of $\pi|A_0$ are zero-dimensional. We also have $(b,b') \in A_0$. The reason is the following: each set $\overline{\pi^{-1}((L_j \cap U) \setminus S)}$ has dimension $(n-1)$ and contains a component that passes through (b^j, b'^j) and projects surjectively onto L_j. The limit of these components gives a component of A_0 that passes through (b,b') and with surjective projection to L_0. Let $\tilde{A}_0$ be one of the irreducible components of A_0 that arise this way. Let $\tilde{S}$ be the set of points in L_0 with fibres of positive dimension in $\tilde{A}_0$. Then $\tilde{A}_0$ satisfies the assumptions of the induction hypothesis.

To see this, first note that $\tilde{A}_0$ is a meromorphic correspondence from L_0 to $\mathbb{CP}^N$ with proper surjective projection $\tilde{\pi} : \tilde{A}_0 \to L_0 \cap U$.

Second, if $z \in (L_0 \cap M \cap U) \setminus \tilde{S}$, then $\tilde{\pi}' \circ \tilde{\pi}^{-1}(z) \subset K$. This is evident if $z \in (L_0 \cap M \cap U) \setminus S$ by the assumptions on A in the theorem and the observation that $\tilde{A}_0 \subset A$. Note that $\tilde{S} \subset L_0 \cap S$, and it therefore suffices to consider the fibre in $\tilde{A}_0$ over points in $(L_0 \cap M \cap S) \setminus \tilde{S}$. Over such points the fibres in $\tilde{A}_0$ are discrete and hence $\tilde{A}_0$ defines a local correspondence given by $\tilde{\pi}' \circ \tilde{\pi}^{-1}$ near a given point in $(L_0 \cap M \cap S) \setminus \tilde{S}$. But since $\dim S < \dim L_0$, $(L_0 \setminus S) \cap M$ is dense in $(L_0 \cap M \cap S)$. In particular, given $p \in (L_0 \cap M \cap S) \setminus \tilde{S}$, there is a sequence $p^j \in (L_0 \setminus S) \cap M$ that converges to p. By assumption, the entire fibre in A (and therefore in $\tilde{A}_0$) over p^j is contained in K and since $\tilde{A}_0$ is a locally a correspondence over p, it follows that the fibre in $\tilde{A}_0$ over p is also contained in K by the continuity of the discrete fibres.

Thus, by the induction hypothesis, $\tilde{S} \cap (M \cap L_0) = \varnothing$. This means that the meromorphic correspondence defined by $\tilde{A}$ is holomorphic, and so $(b,b') \in K$. This proves the theorem.

$\square$

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